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Constraint dynamics and gravitons in three dimensions

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Abstract: The complete non-linear three-dimensional Einstein gravity with gravitational Chern-Simons term and cosmological constant are studied in dreibein formulation. The constraints and their algebras are computed in an explicit form. From counting the number of first and second class constraints, the number of dynamical degrees of freedom, which equals to the number of propagating graviton modes, is found to be 1, *regardless of* the value of cosmological constant. I note also that the usual equivalence with Chern-Simons gauge theory does *not* work for general circumstances.

Keywords: Models of Quantum Gravity, Chern-Simons Theories.

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Contents

1. Introduction

In three-dimensional spacetime, there is no propagating (i.e., dynamical) degrees of freedom in the bulk with either Einstein-Hilbert action or gravitational Chern-Simons term (GCS) with coefficient $1/\mu$. However, it is known that the combined action with a *vanishing* cosmological constant, which is known as "topologically massive gravity" in the literatures, has a *single* propagating, massive, spin-2 mode [1]. Recently, there have been several works toward the generalization with a negative cosmological constant $\Lambda = -1/l^2$ [2, 3]. (For some earlier related works, see refs. [4, 5].) But the results do not seem to be in consensus completely. In ref. [2], the wave function for the gravitons and their corresponding energies are computed for the *linearized* excitations. And it is argued that the theory is unstable/inconsistent for generic values of μ due to "negative" energies for the massive gravitons. However at the critical value $\mu l = 1$, the massive gravitons "disappear" due to vanishing energies. (See ref. [6] for a supporting analysis.) In ref. [3], the linearized excitations of the gravitons as well as the scalar and photons are studied in the "light-front" coordinates and it is argued that the massive graviton modes can not be gauged away at the critical value of μ. (See also ref. [7] for a concurrent analysis.) Rather, at the critical value, it is found that the linearized topologically massive gravity is equivalent to "topologically massive electrodynamics" with a mass parameter $\mu_{E/M} = 2$. And also, the computations show some splitting of the masses for the gauge invariant fields even though there is just one independent degrees of freedom.

On the other hand, it is also known that the three-dimensional (anti-) de Sitter gravity with or without the GCS term can be written as a Chern-Simons gauge theory, which does not have the dynamical degrees of freedom in the bulk [8]. This seems to be obviously contradict to the existence of the gravitons in refs. [2, 3].

In this paper, I consider the constraint algebras in the fully non-linear theory in dreibein formulation. From counting the number of first and second class constraints, I found that the number of independent degrees of freedom, which equals to the number of propagating graviton modes, is 1, *regardless of* the values of cosmological constant. I do not see any evidence of the disappearing degrees of freedom at the critical value $\mu l = 1$ and this *seems* to support the argument of ref. [3]. But, I note that there is a puzzling feature in this result. I note also that the usual equivalence with Chern-Simons gauge theory does *not* work for general circumstances.

2. Hamiltonian formulation

In this section, I consider the Hamiltonian formulation of the topologically massive gravity with a cosmological constant $\Lambda = -1/l^2$, in dreibein formulation [1, 4, 5, 8−11]. The action on a manifold $\mathcal{M}$, omitting some possible boundary terms, is given by

$$I = -\frac{1}{16\pi G}\int_{\mathcal{M}}\left[2e^a\wedge R_a + \frac{1}{3l^2}\epsilon_{abc}e^a\wedge e^b\wedge e^c + \frac{1}{\mu}\omega^a\wedge\left(d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^b\wedge\omega^c\right) + \lambda^a\wedge T_a\right] \tag{2.1}$$

in form notation with the dreibein and spin-connection 1-*forms* $e^a = e^a_\mu dx^\mu$, $\omega^a = \omega^a_\mu dx^\mu$, respectively.[1] The first and the second terms are the conventional Einstein-Hilbert and the cosmological constant terms, respectively, with the curvature $R_a = d\omega_a + (1/2)\epsilon_{abc}\omega^b\wedge\omega^c$. The last term is introduced in order to consider the zero-torsion condition

$$T_a \equiv de_a + \epsilon_{abc}\omega^b\wedge e^c = 0 \tag{2.2}$$

with an auxiliary field λ^a in which λ^a_i is *dynamical* because it multiplies a velocity $\dot{e}_{ai}$.[2] I have chosen the sign in front of the Einstein-Hilbert part (with positive Newton's constant G) in agreement with the usual convention in anti-de Sitter space [2, 4, 5, 8−11, 14, 15] and all other gravity theories in higher dimensions [16, 17], but opposite to the original formulation without cosmological constant [1, 4] and ref. [3]. The reason for this choice is that its black hole solution in the $\mu\to\infty$ limit, i.e., Einstein-Hilbert limit, can be sensible, i.e., having "positive" black hole mass, only with this sign choice.[3]

The first-order formulation of the action (2.1) is given by

$$I = \int_{\mathcal{M}} d^3x[\pi^{ai}\dot{e}_{ai} + \Pi^{ai}\dot{\omega}_{ai} + P^{ai}\dot{\lambda}_{ai} - e^a_0\mathcal{H}_a - \omega^a_0\mathcal{K}_a - \lambda^a_0\mathcal{T}_a - \partial_i\gamma^i] \tag{2.3}$$

[1] The Greek letters $(\mu,\nu,\cdots)$ denote the space-time indices and Latin $(i,j,\cdots)$ denote the *space* indices. Latin $(a,b,\cdots)$ denote the internal Lorentz indices and the indices are raised and lowered by the metric η_{ab}=diag$(-1,1,1)$ (see ref. [11] for more details). I also take the convention $\epsilon_{012} = -\epsilon^{012} = 1$ and $\epsilon^{ij}\equiv\epsilon^{0ij}$.

[2] It seems that there are some dual maps between e^a_μ, ω^a_μ, and λ^a_μ due to the same tensor structure in three dimensions [8]. However, due to the difference in the *internal* Lorentz transformation $\delta e^a = \epsilon^{abc}e_b\theta_c$, $\delta\omega^a = \mathcal{D}\theta^a$ for the infinitesimal parameter θ^a, the fields λ^a and e^a would transform differently also in order that the defining action (2.1) be invariant under the transformation. So, the physical contents would be completely different under the dual maps [12, 13]

[3] In the absence of the cosmological constant, there is no a priori reason to fix the sign since there are no gravitons which can mediate the interactions between massive particles [18]. The sign is significant only when the Chern-Simons interaction of (2.1) is introduced. The positivity of the gravitational energy [18] and the attractiveness of the gravitational interaction [19] depend crucially on the overall sign.

with the conjugate momenta π^{ai}, Π^{ai}, P^{ai} for e_{ai}, ω_{ai}, λ_{ai}, respectively, and ($\bar{\epsilon}^{ij} \equiv \epsilon^{ij}/16\pi G$)

$$\begin{aligned}
\mathcal{H}_a &= \bar{\epsilon}^{ij}\left[R_{aij} + \frac{1}{l^2}\epsilon_{abc}e_i^b e_j^c - 2\epsilon_{abc}\lambda_i^b\omega_j^c + 2\partial_j\lambda_{ai}\right], \\
\mathcal{K}_a &= \bar{\epsilon}^{ij}\left[-\frac{1}{\mu}R_{aij} + T_{aij} - 2\epsilon_{abc}\lambda_i^b e_j^c\right], \\
\mathcal{T}_a &= -\bar{\epsilon}^{ij}T_{aij}, \\
\gamma^i &= -\bar{\epsilon}^{ij}\left[e_i^a\omega_{a0} - \frac{1}{\mu}\omega_i^a\omega_{a0} - 2\lambda_i^a e_{a0}\right].
\end{aligned} \tag{2.4}$$

The Poisson brackets among the canonical variables are given by

$$\{e_i^a(x), \pi_b^j(y)\} = \{\omega_i^a(x), \Pi_b^j(y)\} = \{\lambda_i^a(x), P_b^j(y)\} = \delta_b^a\delta_i^j\delta^2(x-y). \tag{2.5}$$

3. Constraint algebras and number of degrees of freedom

The *primary* constraints of the action (2.1) are given by

$$\begin{aligned}
&\Phi_a^0 \equiv \pi_a^0 \approx 0, \qquad \Phi_a^i \equiv \pi_a^i - 2\bar{\epsilon}^{ij}\lambda_{aj} \approx 0, \\
&\Psi_a^0 \equiv \Pi_a^0 \approx 0, \qquad \Psi_a^i \equiv \Pi_a^i + \bar{\epsilon}^{ij}\left(2e_{aj} - \frac{1}{\mu}\omega_{aj}\right) \approx 0, \\
&\Gamma_a^\mu \equiv P_a^\mu \approx 0,
\end{aligned} \tag{3.1}$$

from the canonical definition of conjugate momenta, $\pi_a^\mu \equiv \delta I/\delta\dot{e}_\mu^a$, $\Pi_a^\mu \equiv \delta I/\delta\dot{\omega}_\mu^a$, $P_a^\mu \equiv \delta I/\delta\dot{\lambda}_\mu^a$. Here, the weak equality '$\approx$' means that the constraint equations are used only after working out a Poisson bracket. The conservation of the constraints $C^0 \equiv (\Phi_a^0, \Psi_a^0, \Gamma_a^0)$, i.e., $\dot{C}^0 = \{C^0, H_C\} \approx 0$, which is a consistency condition, with the canonical Hamiltonian

$$H_C = \int d^2x[e_0^a\mathcal{H}_a + \omega_0^a\mathcal{K}_a + \lambda_0^a\mathcal{T}_a + \partial_i\gamma^i] \tag{3.2}$$

produces the *secondary* constraints,

$$\mathcal{H}_a \approx 0, \quad \mathcal{K}_a \approx 0, \quad \mathcal{T}_a \approx 0. \tag{3.3}$$

With the primary and secondary constraints (3.1) and (3.3), I consider the *extended* Hamiltonian which can accommodate the arbitrariness in the equations of motions due to the constraints:

$$H_E = H_C + \int d^2x[u_\mu^a\Phi_a^\mu + v_\mu^a\Psi_a^\mu + z_\mu^a P_a^\mu]. \tag{3.4}$$

The coefficients u_μ^a, v_μ^a, z_μ^a are determined as follows, by considering the consistency conditions, $\dot{C}^i = \{C^i, H_E\} \approx 0$ with $C^i \equiv (\Phi_a^i, \Psi_a^i, \Gamma_a^i)$:

$$\begin{aligned}
u_{ai} &= \partial_i e_{0a} - \epsilon_{abc}(e_0^b\omega_i^c + \omega_0^b e_i^c), \\
v_{ai} &= \partial_i\omega_{0a} - \epsilon_{abc}\omega_0^b\omega_i^c - \mu\epsilon_{abc}(e_0^b\lambda_i^c + \lambda_0^b e_i^c), \\
z_{ai} &= \partial_i\lambda_{0a} - \epsilon_{abc}[\lambda_0^b(\omega_i^c + \mu e_i^c) + (\omega_0^b + \mu e_0^b)\lambda_i^c] + \frac{1}{l^2}\epsilon_{abc}e_0^b e_j^c.
\end{aligned} \tag{3.5}$$

The extended Hamiltonian (3.4) reads then as

$$H_E = \int d^2x[e_0^a\bar{\mathcal{H}}_a + \omega_0^a\bar{\mathcal{K}}_a + \lambda_0^a\bar{\mathcal{T}}_a + u_0^a\Phi_a^0 + v_0^a\Psi_a^0 + z_0^aP_a^0 + \partial_i\bar{\gamma}^i] \tag{3.6}$$

with modified constraints,

$$\begin{aligned}
\bar{\mathcal{H}}_a &\equiv \mathcal{H}_a - \mathcal{D}_i\Phi_a^i - \mu\epsilon_{abc}\lambda_i^b\Psi^{ci} + \epsilon_{abc}\left(-\mu\lambda_i^b + \frac{1}{l^2}e_i^b\right)P^{ci} \approx 0, \\
\bar{\mathcal{K}}_a &\equiv \mathcal{K}_a - \epsilon_{abc}e_i^b\Phi^{ci} - \mathcal{D}_i\Psi_a^i - \epsilon_{abc}\lambda_i^bP^{ci} \approx 0, \\
\bar{\mathcal{T}}_a &\equiv \mathcal{T}_a - \mu\epsilon_{abc}e_i^b\Psi^{ci} - \mathcal{D}_iP_a^i - \mu\epsilon_{abc}e_i^bP^{ci} \approx 0,
\end{aligned} \tag{3.7}$$

and the covariant derivatives $(\mathcal{D}_i)^c{}_a = \delta_a^c\partial_i + \epsilon^c{}_{ab}\omega_i^b$. After a tedious but straightforward computation I get

$$\begin{aligned}
\{\Phi_a^i(x), \Psi_b^j(y)\} &= 2\bar{\epsilon}^{ij}\eta_{ab}\delta^2(x-y), \\
\{\Phi_a^i(x), P_b^j(y)\} &= -2\bar{\epsilon}^{ij}\eta_{ab}\delta^2(x-y), \\
\{\Phi_a^i(x), \bar{\mathcal{H}}_b(y)\} &= \frac{1}{l^2}\epsilon_{abc}P^{ci}\delta^2(x-y), \\
\{\Phi_a^i(x), \bar{\mathcal{K}}_b(y)\} &= -\epsilon_{abc}\Phi^{ci}\delta^2(x-y), \\
\{\Phi_a^i(x), \bar{\mathcal{T}}_b(y)\} &= -\mu\epsilon_{abc}(\Psi^{ci} + P^{ci})\delta^2(x-y), \\
\{\Psi_a^i(x), \Psi_b^j(y)\} &= -\frac{2}{\mu}\bar{\epsilon}^{ij}\eta_{ab}\delta^2(x-y), \\
\{\Psi_a^i(x), \bar{\mathcal{H}}_b(y)\} &= -\epsilon_{abc}\Phi^{ci}\delta^2(x-y), \\
\{\Psi_a^i(x), \bar{\mathcal{K}}_b(y)\} &= -\epsilon_{abc}\Psi^{ci}\delta^2(x-y), \\
\{\Psi_a^i(x), \bar{\mathcal{T}}_b(y)\} &= -\epsilon_{abc}P^{ci}\delta^2(x-y), \\
\{P_a^i(x), \bar{\mathcal{H}}_b(y)\} &= -\mu\epsilon_{abc}(\Psi^{ci} + P^{ci})\delta^2(x-y), \\
\{P_a^i(x), \bar{\mathcal{K}}_b(y)\} &= -\epsilon_{abc}P^{ci}\delta^2(x-y), \\
\{\bar{\mathcal{H}}_a(x), \bar{\mathcal{H}}_b(y)\} &\approx \{\mathcal{H}_a(x), \bar{\mathcal{H}}_b(y)\} = \left[\frac{1}{l^2}\epsilon_{abc}\mathcal{T}^c - 2\mu\bar{\epsilon}^{ij}\lambda_{ai}\lambda_{bj}\right]\delta^2(x-y), \\
\{\bar{\mathcal{H}}_a(x), \bar{\mathcal{K}}_b(y)\} &\approx \{\mathcal{H}_a(x), \bar{\mathcal{K}}_b(y)\} = -\epsilon_{abc}\mathcal{H}^c\delta^2(x-y), \\
\{\bar{\mathcal{H}}_a(x), \bar{\mathcal{T}}_b(y)\} &\approx \{\mathcal{H}_a(x), \bar{\mathcal{T}}_b(y)\} = [-\mu\epsilon_{abc}(\mathcal{K}^c + \mathcal{T}^c) + 2\mu\bar{\epsilon}^{ij}\lambda_{ai}e_{bj}]\delta^2(x-y), \\
\{\bar{\mathcal{K}}_a(x), \bar{\mathcal{K}}_b(y)\} &\approx \{\mathcal{K}_a(x), \bar{\mathcal{T}}_b(y)\} = -\epsilon_{abc}\mathcal{K}^c\delta^2(x-y), \\
\{\bar{\mathcal{K}}_a(x), \bar{\mathcal{T}}_b(y)\} &\approx \{\mathcal{K}_a(x), \bar{\mathcal{T}}_b(y)\} = \epsilon_{abc}\mathcal{T}^c\delta^2(x-y), \\
\{\bar{\mathcal{T}}_a(x), \bar{\mathcal{T}}_b(y)\} &= [-2\mu\bar{\epsilon}^{ij}e_{ai}e_{bj} - \mu(e_{ai}P_b^i - e_{bi}P_a^i)]\delta^2(x-y).
\end{aligned} \tag{3.8}$$

Using the above constraint algebras, one can easily see that there are *thirdary* constraints from the consistencies of $\bar{\mathcal{H}}_a \approx 0$, $\bar{\mathcal{T}}_a \approx 0$ constraints (no additional constraints from $\bar{\mathcal{K}}_a \approx 0$)

$$\begin{aligned}
\dot{\bar{\mathcal{H}}}_a(x) &= \{\bar{\mathcal{H}}_a(x), H_E\} \\
&\approx -2\mu\bar{\epsilon}^{ij}\lambda_{ai}(e_0^b\lambda_{bj} - \lambda_0^b e_{bj}) \equiv \Sigma_a \approx 0, \\
\dot{\bar{\mathcal{T}}}_a(x) &= \{\bar{\mathcal{T}}_a(x), H_E\} \\
&\approx 2\mu\bar{\epsilon}^{ij}e_{ai}(e_0^b\lambda_{bj} - \lambda_0^b e_{bj}) \equiv \chi_a \approx 0.
\end{aligned} \tag{3.9}$$

The additional constraints Σ_a, χ_a have the following non-vanishing brackets:

$$\begin{aligned}
\{\Sigma_a(x), \Phi_b^0(y)\} &= -2\mu\bar{\epsilon}^{ij}\lambda_{ai}\lambda_{bj}\delta^2(x-y), \\
\{\Sigma_a(x), \Phi_b^i(y)\} &= -2\mu\bar{\epsilon}^{ij}\lambda_{aj}\lambda_{b0}\delta^2(x-y), \\
\{\Sigma_a(x), P_b^0(y)\} &= 2\mu\bar{\epsilon}^{ij}\lambda_{ai}e_{bj}\delta^2(x-y), \\
\{\Sigma_a(x), P_b^i(y)\} &= -2\mu\bar{\epsilon}^{ij}[\eta_{ab}(e_0^c\lambda_{cj} - \lambda_0^c e_{cj}) - \lambda_{aj}e_{b0}]\delta^2(x-y), \\
\{\chi_a(x), \Phi_b^0(y)\} &= 2\mu\bar{\epsilon}^{ij}e_{ai}e_{bj}\delta^2(x-y), \\
\{\chi_a(x), \Phi_b^i(y)\} &= 2\mu\bar{\epsilon}^{ij}[\eta_{ab}(e_0^c\lambda_{cj} - \lambda_0^c e_{cj}) + e_{aj}\lambda_{b0}]\delta^2(x-y), \\
\{\chi_a(x), P_b^0(y)\} &= -2\mu\bar{\epsilon}^{ij}e_{ai}e_{bj}\delta^2(x-y), \\
\{\chi_a(x), P_b^i(y)\} &= -2\mu\bar{\epsilon}^{ij}e_{aj}e_{b0}\delta^2(x-y).
\end{aligned} \tag{3.10}$$

Further investigations of the consistency conditions for the constraints Σ_a, χ_a, i.e., $\{\Sigma_a, H_E'\} \approx 0$, $\{\chi_a, H_E'\} \approx 0$ with $H_E' = H_E + \int d^2x(\alpha^a\Sigma_a + \beta^a\chi_a)$ do not yield new constraints but determine the coefficients u_0^a, z_0^a, α^a, and β^a. This completes Dirac's consistency procedure for finding the complete set of constraints. Although the algebras are complicated nevertheless one can see that the constraints Ψ_a^0, $\bar{\mathcal{K}}_a$ are *first* class and the constraints Φ_a^μ, P_a^μ, Ψ_a^i, Σ_a, χ_a, $\bar{\mathcal{H}}_a$, $\bar{\mathcal{T}}_a$ are *second* class. Here, one might consider some special configurations of $\lambda_{a\mu}$ and $e_{a\mu}$, i.e., $A_{ab} \equiv 2\bar{\epsilon}^{ij}\lambda_{ai}\lambda_{bj} \approx 0$, $B_{ab} \equiv 2\bar{\epsilon}^{ij}\lambda_{ai}e_{bj} \approx 0$, $C_{ab} \equiv 2\bar{\epsilon}^{ij}e_{ai}e_{bj} \approx 0$ (neglecting the trivial configurations of $\lambda_{ai} = e_{ai} = 0$) such that some of these constraints may become first class or dependent (i.e., irregular) [20]. But, this is *not* relevant to our case (for some related discussions, see also [21, 22]): $A_{ab} \approx 0$ or $C_{ab} \approx 0$ implies that $\lambda_{a\mu}$ or $e_{a\mu}$, respectively, is not invertible from the fact of $det(\lambda_{a\mu}) = \epsilon^{abc}\lambda_{a0}A_{bc} \approx 0$, $det(e_{a\mu}) = \epsilon^{abc}e_{a0}C_{bc} \approx 0$, but the invertibility has been implicitly assumed from the construction;[4] moreover, $B_{ab} \approx 0$ would not be generally true since this implies, from (2.4), $\bar{\epsilon}^{ij}R_{aij} = 0$, i.e., pure Einstein gravity solutions, certainly not a restriction I wish to consider.

To compute the number of dynamical degrees of freedom I use the standard formula, at any point x,

$$s = \frac{1}{2}(2n - 2N_1 - N_2), \tag{3.11}$$

where $2n$ is the number of canonical variables, N_1 is the number of first class constraints, and N_2 is the number of second class constraints. Then, according to the above constraint

[4] In quantum theory, the non-invertible $e_{a\mu}$ might be permitted. See ref. [8] for example.

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algebras, I have $n = 9$ [e^a_μ, ω^a_μ, λ^a_μ], $N_1 = 2$, and $N_2 = 12$ for "each internal index a". This represents that the system in terms of the metric $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$[5] has a *single* dynamical degrees of freedom which equals to the number of propagating graviton modes.

Here, in counting the number of degrees of freedom it would be important to check that they possess a well defined spectrum. However, unfortunately the kinematic counting of (3.11) gives *no* information as to their (in)stabilities.

Before finishing this section, several remarks are in order. First, the presence of cosmological constant does not modify second class constraint algebras nor the number of the first class constraints. So, I do not see any evidence of the disappearing degrees of freedom at the critical value $\mu l = 1$: There are two possible scenarios for this effect, i.e., (a) there is an additional "first" class constraint, representing a new symmetry, at the critical value of the cosmological constant [24], (b) there are compensation terms from the cosmological constant in second class constraint algebras and the complete compensation is achieved at the critical value such that the second class constraint becomes first class [25]; but, neither of these "symmetry enhancements" do not occur in the system. This *seems* to support the argument of ref. [3]. Second, the action (2.1) is not equivalent to the Chern-Simons gauge gravity [8, 11], generally. They are equivalent only when one identify $\lambda^a_\mu = e^a_\mu/\mu l^2$ which changes enormously the constraint algebras. Actually, the constraint analysis of this system [26] leads to $n = 6$, $N_1 = 4$, $N_2 = 4$ so that $s = 0$, i.e., *no* dynamical degrees of freedom. This is consistent with the fact that the Chern-Simons gauge theory does not have the dynamical degrees of freedom.[6] On the other hand, if one does not introduce the last torsion term in (2.1) such that the torsion does not vanish anymore, one has the same numbers of $N_1 = 4$, $N_2 = 4$ such that $s = 0$ also.

4. Discussion

I have shown that counting the number of first and second class constraints leads to a "single" dynamical degrees of freedom for the *metric*, regardless of the value of the cosmological constant $\Lambda = -1/l^2$. This seems to support the argument of ref. [3] (and ref. [7] also), but this is rather surprising from the following reasons. First, in the context of the bulk gravity action (2.1), it is known that there are critical values of the cosmological constant $|\mu l| = 1$ [11, 27], where the characters of the BTZ black hole solution and matter fields in the black hole background are dramatically changed. Second, in the context of boundary CFT at the asymptotic infinity also, the structure of the CFT and its Hilbert space changes at the critical value. But, at present, there is no clear understanding of why the fully non-linear constraints do not show up the above critical features [11].

[5] It would be also interesting to consider the constraint algebras in the metric formulation directly [23], where the GCS term is a third-derivative order and Ostrogradsky method is needed [13].

[6] Three-dimensional gravity without cosmological constant and GCS term can be described by $ISO(2,1)$ Chern-Simons gauge theory [8]. However, in the presence of the GCS term, a second invariant quadratic form for the Lie algebra is *degenerate* such that the gauge theory formulation does *not* exist. This is consistent with the existence of a graviton in the system [1].

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